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Corrigendum

Corrigendum to "Limit theorems for iterated random functions by regenerative methods" [Stochastic Process. Appl. 96 (2001) 123–142][☆]

Gerold Alsmeyer[a,*], Cheng-der Fuh[b]

[a] *Institut für Mathematische Statistik, Fachbereich Mathematik, Westfälische Wilhelms-Universität Münster, Einsteinstraße 62, D-48149 Münster, Germany*
[b] *Institute of Statistical Science, Academia Sinica, Taipei, Taiwan, ROC*

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Section 3 of the article contains a number of mistakes. The correct definitions of $\tau(n)$ and C_n are $\tau(n) = \inf\{j \geqslant 0\colon \sigma_j \geqslant n\}$ for $n \geqslant 0$ and

$$C_n \stackrel{\text{def}}{=} \max\{d(F_{\sigma_{n-1}+1:\sigma_n}(x_0), x_0); d(F_{\sigma_{n-1}+1:\sigma_n}(x_0), F_{\sigma_{n-1}+1:k}(x_0)), \sigma_{n-1} < k < \sigma_n\}$$

for $n \geqslant 1$. Note that (3.5) and the subsequent independence assertions hold true only then because $\tau(n)$ is now a stopping time.

In the proof of Lemma 3.2, the correct definition of F_n' reads $F_n' \stackrel{\text{def}}{=} F_{\sigma_{n-1}+1:\sigma_n}$. Then the sequence $(F_{n:1}'(M_0))_{n\geqslant 1}$ is indeed an IFS of i.i.d. Lipschitz maps with backward iterations $\hat{M}_{\sigma_n} = F_{1:n}'(M_0)$, $n \geqslant 1$, but it generally differs from $(M_{\sigma_n})_{n\geqslant 1}$.

The bound for C_1 in (3.11) is wrong. This is the most serious error for it entails that Lemma 3.4 and parts of Theorem 2.3 and Corollary 2.5 must be revised as will be described further below. The correction of (3.11) including the subsequent estimation looks as follows: Put

$$U_n \stackrel{\text{def}}{=} \max\left\{\prod_{j=1}^{k} L_j, 0 \leqslant k \leqslant n\right\}$$

for $n \geqslant 0$ (where $\prod_{k=1}^{0} \stackrel{\text{def}}{=} 1$ as usual) and note that

$$U_n \leqslant \prod_{k=1}^{n}(1 + L_k) \quad \text{and} \quad \log_* U_n \leqslant \sum_{k=1}^{n} \log_* L_k. \tag{3.10}$$

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* Corresponding author. Fax: +49-2518332712.
E-mail address: gerolda@math.uni-muenster.de (G. Alsmeyer).

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Now use

$$
\begin{aligned}
C_1 &\leqslant \sum_{j=1}^{\sigma_1} d(F_{1:j}(x_0), F_{1:j-1}(x_0)) \\
&\leqslant \sum_{j=1}^{\sigma_1} L_1 \cdots L_{j-1} d(F_j(x_0), x_0) \\
&\leqslant U_{\sigma_1} \sum_{j=1}^{\sigma_1} d(F_j(x_0), x_0), \qquad (3.11)
\end{aligned}
$$

(3.10) and Wald's first identity to infer

$$
\begin{aligned}
\mathbb{E} \log_* d(F_{1:\sigma_1}(x_0), x_0) \leqslant \mathbb{E} \log_* C_1 & \\
&\leqslant \mathbb{E}\left(\sum_{j=1}^{\sigma_1} \log_* L_j\right) + \mathbb{E}\left(\sum_{j=1}^{\sigma_1} \log_* d(F_j(x_0), x_0)\right) \\
&= (\mathbb{E} \log_* L_1 + \mathbb{E} \log_* d(F_1(x_0), x_0))\mathbb{E}\sigma_1 < \infty.
\end{aligned}
$$

In the following we summarize the important changes caused by the corrected inequality (3.11).

Proof of Lemma 3.3 (*First paragraph*).
We retain the notation of the proof of the previous lemma. If (3.12) holds, then $\mathbb{E}\sigma_1^{p+1} < \infty$. Using (3.10) and (3.11), we obtain

$$
\mathbb{E} \log_*^{p+1} C_1 \leqslant \mathbb{E}\left(\sum_{j=1}^{\sigma_1} \log_* L_j\right)^{p+1} + \mathbb{E}\left(\sum_{j=1}^{\sigma_1} \log_* d(F_j(x_0), x_0)\right)^{p+1}.
$$

Now each of the terms on the right-hand side is the $(p+1)$-st moment of a stopped sum of i.i.d. random variables. That they are finite follows from Theorem I.5.2 in Gut (1988). □

The biggest changes occur in Lemma 3.4 and its proof. Their revisions are therefore presented in complete form:

Lemma 3.4. *Let* $\gamma \in (0,1)$ *and* $p > 0$. *If*

$$
\mathbb{E} L_1^p < \infty \quad \textit{and} \quad \mathbb{E} d(F_1(x_0), x_0)^p < \infty, \qquad (3.20)
$$

then the following assertions hold for some $\eta > 0$:

$$\mathbb{E}C_1^{2\eta} < \infty, \tag{3.21}$$

$$\mathbb{E}D_0^{2\eta} = \mathbb{E}D_{\tau(n)}^{2\eta} < \infty \tag{3.22}$$

for all $n \geqslant 0$. *The family* $\{C_{\tau(n)}^{\eta}, n \geqslant 0\}$ *is uniformly integrable and satisfies*

$$\sup_{n \geqslant 0} \mathbb{E}C_{\tau(n)}^{\eta} \leqslant \mathbb{E}\sigma_1 C_1^{\eta} < \infty. \tag{3.23}$$

Moreover, the first condition of (3.20) *implies*

$$\mathbb{P}(L_{1:n} > \varepsilon\alpha^n) \leqslant \varepsilon^{-1}\alpha^n \tag{3.24}$$

for all $n \geqslant 1$, $\varepsilon > 0$ *and a suitable* $\alpha \in (0,1)$.

It is to be noted that, by (3.5), the $D_{\tau(n)}$ are identically distributed whence (3.14) and (3.22) trivially imply the uniform integrability of $\{\log^p(1 + D_{\tau(n)}), n \geqslant 0\}$ and $\{D_{\tau(n)}^{2\eta}, n \geqslant 0\}$, respectively.

Proof. If (3.20) holds, which in particular means that $\log_* L_1$ has an exponential moment, then σ_1 has an exponential moment, too. Hence, a standard argument shows that we can find a sufficiently small $\eta \leqslant p/4$ such that

$$\mathbb{E}\exp\left(4\eta \sum_{k=1}^{\sigma_1} \log_* L_k\right) < \infty. \tag{3.25}$$

It follows from (3.10) and (3.11) that

$$C_1^{2\eta} \leqslant \exp\left(2\eta \sum_{j=1}^{\sigma_1} \log_* L_j\right) \left(\sum_{k=1}^{\sigma_1} d(F_k(x_0), x_0)\right)^{2\eta} \quad \text{a.s.}$$

and thus with Hölder's inequality

$$\mathbb{E}C_1^{2\eta} \leqslant \left(\mathbb{E}\exp\left(4\eta \sum_{j=1}^{\sigma_1} \log_* L_j\right)\right)^{1/2} \left(\mathbb{E}\left(\sum_{k=1}^{\sigma_1} d(F_k(x_0), x_0)\right)^{4\eta}\right)^{1/2}.$$

The first expectation on the right-hand side is finite by (3.25), while this holds for the second as being the expectation of a stopped sum of i.i.d. random variables with finite moments of order $4\eta \leqslant p$ (see Gut, 1988).

(3.22), in case $2\eta \geqslant 1$, follows immediately by using (3.6) and the infinite version of Minkowski's inequality. They give

$$(\mathbb{E}D_0^{2\eta})^{1/2\eta} \leqslant \sum_{j \geqslant 1} \gamma^{j-1} (\mathbb{E}C_1^{2\eta})^{1/2\eta} = \frac{(EC_1^{2\eta})^{1/2\eta}}{1-\gamma} < \infty.$$

If $0 < 2\eta < 1$, then $t \mapsto t^{2\eta}$ is subadditive and thus

$$\mathbb{E}D_0^{2\eta} \leqslant \sum_{j \geqslant 1} \gamma^{2\eta(j-1)} \mathbb{E}C_1^{2\eta} \leqslant \frac{\mathbb{E}C_1^{2\eta}}{1-\gamma^{2\eta}} < \infty.$$

As to the proof of (3.23), note first that $\mathbb{E}C_1^{2\eta} < \infty$ and $\mathbb{E}e^{s\sigma_1} < \infty$ for some $s > 0$ imply $\mathbb{E}C_1^{\eta}\sigma_1 < \infty$. The assertion now follows because (3.19) with $H(t) = t^{\eta}$ yields

$$\sup_{n\geqslant 0} \mathbb{E}C_{\tau(n)}^{\eta} \leqslant \mathbb{E}\sigma_1 C_1^{\eta}.$$

The proof of (3.24) remains unaltered. $\square$

Due to the weaker form of Lemma 3.4 we also have to revise Theorem 2.3 and the related Corollary 2.5. Changes occur only in 2.3(b),(d) and in 2.5(b). The assertions in 2.3(b) and 2.5(b) contained another mistake which is now also corrected. The proof of Theorem 2.3 in Section 4 is essentially the same as before when replacing p with η in the obvious places. In addition to that replace the key inequality in the proof of 2.3(b) by

$$(1 + d(x,x_0))^{-q} d(\hat{M}_{\infty}^{x_0}, \hat{M}_n^{x})^q \leqslant \gamma^{q(\tau(n)-1)}(C_{\tau(n)} + D_{\tau(n)})^q + L_{1:n}^q \quad \text{a.s.}$$

and recall from the end of the proof of Lemma 3.4 that $\mathbb{E}L_{1:n}^q = \beta_q^n$ for some $\beta_q \in (0,1)$ and all $q \in (0,\eta)$, $\eta > 0$ sufficiently small.

Theorem 2.3. *Given the situation of Theorem* 2.1 *and additionally condition* (1.9) *for some* $p > 0$, *the following assertions hold*:

(a) *For each* $\gamma \in (\gamma^*, 1)$,

$$\lim_{n\to\infty} \alpha_{\gamma}^{-n} \mathbb{P}_x(d(\hat{M}_{\infty}, \hat{M}_n) > \gamma^n) = 0$$

for all $x \in \mathbb{X}$ *and some* $\alpha_{\gamma} \in (0,1)$.

(b) *There exists* $\eta > 0$ *such that for each* $q \in (0,\eta)$,

$$\lim_{n\to\infty} \sup_{x\in\mathbb{X}} \alpha_q^{-n}(1 + d(x,x_0))^{-q} \mathbb{E}_x d(\hat{M}_{\infty}, \hat{M}_n)^q = 0$$

for some $\alpha_q \in (0,1)$. *The same holds true for* $q = \eta$ *with* $\alpha_q = 1$.

(c) $d(P^n(x,\cdot),\pi) \leqslant A_x r^n$ *for all* $n \geqslant 0$, *some* $r \in (0,1)$ *and a constant* A_x *of the form* $\max\{A, d(x,x_0)\}$. *The constants r and A do not depend on x nor n*.

(d) $\int_{\mathbb{X}} d(x,x_0)^{\eta}\, \pi(\mathrm{d}x) = \int_0^{\infty} \eta t^{\eta-1} \pi(x : d(x,x_0) > t)\, \mathrm{d}t < \infty$ *for some* $\eta > 0$.

Corollary 2.5. *Given the situation of Theorem* 2.3, *the following assertions hold*:

(a) *For each* $\gamma \in (\gamma^*, 1)$,

$$\lim_{n\to\infty} \alpha_{\gamma}^{-n} \mathbb{P}(d(M_n^x, M_n^y) > \gamma^n) = 0$$

for all $x, y \in \mathbb{X}$ *and some* $\alpha_{\gamma} \in (0,1)$.

(b) *There exists* $\eta > 0$ *such that for each* $q \in (0,\eta)$,

$$\lim_{n\to\infty} \sup_{x,y\in\mathbb{X}} \alpha_q^{-n}(1 + d(x,x_0) \vee d(y,x_0))^{-q} \mathbb{E}_x d(M_n^x, M_n^y)^q = 0$$

for some $\alpha_q \in (0,1)$. *The same holds true for* $q = \eta$ *with* $\alpha_q = 1$.

As three minor corrections we finally note that (3.6) should read

$$D_n \leqslant \sum_{j\geqslant 1} \gamma^{j-1} C_{n+j}$$

for all $n \geqslant 0$, that (see two lines below) $C_n = d(F_n(x_0), x_0)$ in the strictly contractive case and that the first equation leading to (3.18) should read

$$\mathbb{P}(H(C_{\tau(n)}) > t) = \sum_{j=1}^{n} U(\{n-j\})\mathbb{P}(\sigma_1 \geqslant j, H(C_1) > t).$$

Reference

Gut, A., 1988. Stopped Random Walks: Limit Theorems and Applications. Springer, New York.